

Slow oscillations in an ocean of varying depth Part 1. Abrupt topography

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This paper is part of a study of quasigeostrophic waves, which depend on the topography of the ocean floor and the curvature of the earth.

In a homogeneous, β -plane ocean, motion of the fluid across contours of constant f/h releases relative vorticity (f is the Coriolis parameter and h the depth). This well-known effect provides a restoring tendency for either Rossby waves (with h constant) or topographic waves over a slope. The long waves in general obey an elliptic partial differential equation in two space variables. Because the equation has been integrated in the vertical direction, the exact inviscid bottom boundary condition appears in variable coefficients.

When the depth varies in only one direction the equation is separable at the lowest order in ω , the frequency upon f . With a simple slope, $|\nabla h/h| = \text{constant}$, the transition from Rossby to topographic waves occurs at $|\nabla h| \sim h/R_e$, where R_e is the radius of the earth. Isolated topographic features are considered in §2. It is found that a step of fractional height δ on an otherwise flat ocean floor reflects the majority of incident Rossby waves when $\delta > 2\omega$. In the ocean ω is usually small, due to continental barriers, so even slight depth variations are important. A narrow *ridge* does not act as a great obstruction but calculations show, for example, that the Mid-Atlantic Ridge is broad enough to reflect all but the lowest mode Rossby waves in the North Atlantic.

Besides isolating oceanic plains from one another, steps and ridges support trapped topographic waves of greatest frequency $\sim \delta/2$, analogous to the potential well solutions in quantum mechanics. These waves cannot carry energy along *abrupt* topography, but they disperse more rapidly over broader slopes; the phase and group speeds may be hundreds of cm/sec. The continental-shelf waves found by Robinson are an example of the latter case. There are many such wave guides, where the f/h contours are crowded, in the deep ocean.

The theory suggests that measurement of Rossby waves will rarely be possible at the coast of a continent.

1. Introduction and derivation

The atmosphere influences the ocean over a broad range of frequencies, from those of surface waves to the slower variations of weather and the seasons. It has long been realized that an ideal ocean responds differently to weather than to

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winds that blow without change forever. Hough (1898) showed that the long-period solutions of Laplace's tidal equation for a layer of fluid on a rotating sphere are distinct from the more rapid gravity waves. The former, 'second-class' motions are due mainly to the rotation. As far as these linear solutions are valid, they may describe the effects of weather on the ocean; indeed, the weather itself is to some extent motion of the same class.

Longuet-Higgins, in a series of papers (1964, 1965*a, b*) has given a thorough treatment of these motions (also known as quasigeostrophic, Rossby, or planetary waves) on a sphere. He has shown the nature of the superposition that accounts for simple lateral boundaries.

The problem of a fluid in a shaped, rotating basin is somewhat similar. Ball (1963*a, b*, 1965*a, b*), has brought to an elegant state the inviscid theory for an elliptic paraboloid (the emphasis in most other rotating basin problems has been on the 'first-class' inertial-gravity waves). The spin-up problems of Greenspan & Howard (1963), Greenspan (1964, 1965) and Holton (1965) are in some ways directed to the same end. Ekman-layer suction may bring the fluid to the speed of its boundaries in a simple manner, but often waves are excited of which the second-class modes are a subset.

The model oceans considered previously have had smooth and simply shaped boundaries. The lack of firm evidence of Rossby waves leads us to include an irregular bottom (the importance of complex coastal boundaries is also worthy of study).

In this paper we consider abrupt topography, finding its effect on Rossby waves over an otherwise featureless bottom, and describing the new oscillations that it induces. In a later paper a continuously rough bottom will be included, and modifications due to density stratification discussed.

The ideas of potential vorticity and the β -plane approximation provide a feel for the motions. In the absence of non-conservative forces, fluid in a thin layer on a sphere satisfies

$$D\chi/Dt = 0, \quad \chi = (f + \zeta)/h,$$

or

$$\frac{D}{Dt} \left(\frac{\zeta}{h} \right) + (\mathbf{u} \cdot \nabla) \frac{f}{h} = 0$$

away from the equator. ζ , h and f are the radial (vertical) components of vorticity, depth and twice the earth's rotation vector, $\mathbf{\Omega}$, respectively, \mathbf{u} is the horizontal velocity, and D/Dt includes the horizontal advection. Large scale, low frequency motions involve coherent columns of fluid with axes parallel to $\mathbf{\Omega}$ (although for a thin layer the difference between Taylor columns parallel to $\mathbf{\Omega}$, and radial, is insignificant at mid-latitudes). Slight gravitational stability in the fluid suppresses the horizontal component of $\mathbf{\Omega}$, however. The columns then are radial; a layer for which $H/L \ll f/N \ll 1$ where N is the Brunt-Väissälä frequency, and L the horizontal scale, is otherwise ignorant of the stratification.

If now the spherical shell is mapped locally onto a plane, with Cartesian coordinates (x, y) oriented east and north, respectively, the dominant effect of sphericity away from high latitudes is in the y -dependence of f , and we may neglect the convergence of the meridians for waves much shorter than the earth's circumference. Rossby (1939) introduced this great simplification, which has

since been examined thoroughly by Veronis (1963), Rattray & Charnell (1966) and Longuet-Higgins (1964 *et seq.*).

From this simple expression of the problem we can see that variations in either f or h provide gradients of χ with the fluid at rest; then movement through changes either in depth or in latitude releases relative vorticity, altering the motion. The similarity of the second-class modes on a sphere and in a basin has been exploited by many workers, for example von Arx (1957), Fultz & Kaylor (1959), Phillips (1965), Pedlosky & Greenspan (1967) and Beardsley (1968), and provided striking laboratory demonstrations of oceanic and atmospheric phenomena.

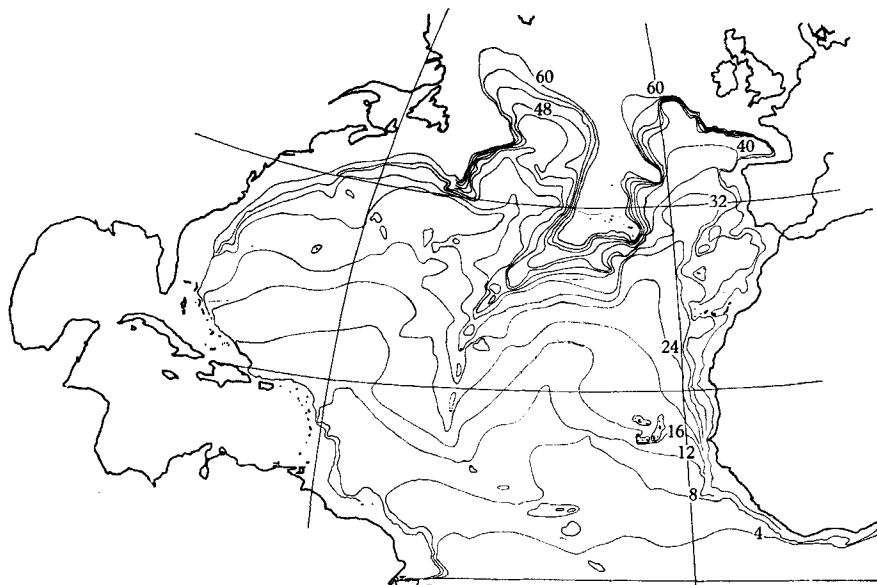


FIGURE 1. Contours of f/h ('isostrophes') for the North Atlantic. The contours are separated by 4° of latitude at $h = 4000$ m (the numbers refer to this base). They are thus slightly bunched together in the north. Many closed contours, and all contours north of 60° N, 4000 m, have been omitted.

The place of second-class motions over a slope in the general scheme of rotating fluids was clarified when Phillips (1965) showed that they are simply a shallow-water subset of the inertial waves that cover the spectrum $\omega = \sigma/f \leq 1$ (σ is the radian frequency).

The idea of geostrophy is fundamental to this type of motion. If variations with time and space are gradual enough ($\omega \ll 1$ and ϵ , the Rossby number, $\equiv U/fL \ll 1$, where U and L are typical horizontal velocity and length scales, respectively) the momentum equation is dominated by the 'geostrophic' balance between pressure gradients and the Coriolis force. The slight departures we are interested in are known as quasigeostrophy. Most of the present work falls under this heading since $\omega \sim L/R_e$ for Rossby waves, where R_e is the earth's radius, and this is usually small because of continental barriers. It will be shown that waves due to small departures from a uniform depth also satisfy $\omega \ll 1$.

If we set $\omega = 0$ and say that ϵ is small (care is dictated, since the limits $\omega \rightarrow 0$, $\epsilon \rightarrow 0$ are singular), the above equations show that streamlines lie close to contours of f/h . This solution was given by Lamb (1932, p. 333), and may describe much of the deep, steady circulation pattern of the ocean.

Our interest is in the linearized waves for which $\omega \ll 1$ and $\epsilon/\omega \ll 1$. When the depth is constant the contours of the restoring agent are latitude lines, but in the real ocean contours of f/h , which might be called 'isostrophes', are badly distorted. These are plotted in figure 1 for the North Atlantic.

The chart was prepared from a Russian topographic map, published in 1963. It was severely averaged and many 'islands' (closed contours) left out to make clearer the general pattern. Since the 'islands' tend to trap both steady and time-dependent motions, a more accurate chart would be of interest. The dynamical effects of roughness and of large features like the Mid-Atlantic ridge are hinted at; we suspect that many scales of Rossby wave will be altered in character.

We are not alone, since many authors have cited the probable importance of bottom topography. The only paper that deals with the problem, however, appears to be that of Robinson & Stommel (1959), who calculated the first perturbation term for small depth variations. This approach will be discussed in a later paper.

We now derive the appropriate form of the equation, listing in a more orderly fashion the assumptions. The momentum and continuity equations for long-wave motion of a homogeneous, inviscid liquid are, in a rotating system on a β -plane,

$$\left. \begin{aligned} u_t - fv &= -g\eta_x, \\ v_t + fu &= -g\eta_y, \\ \eta_t + (hu)_x + (hv)_y &= 0, \end{aligned} \right\} \quad (1.1)$$

where the pressure, assumed hydrostatic, is

$$p = p_{z=\eta} + \rho g(\eta - z)$$

and $f = f_0 + \beta y$, ρ is the density, g the acceleration due to gravity, u , v , w and x , y , z are the eastward, northward and upward velocities and co-ordinates, respectively, and η is the elevation of the free surface. With time dependence $\propto e^{-i\sigma t}$, elimination of u and v gives

$$(h\eta_x)_x + (h\eta_y)_y + \frac{f}{i\sigma}(\eta_x h_y - \eta_y h_x) - \frac{f^2 - \sigma^2}{g}\eta - \frac{\beta h}{f^2 - \sigma^2} \left[\frac{f^2 + \sigma^2}{i\sigma}\eta_x + 2f\eta_y \right] = 0.$$

With the restriction $\sigma \ll f$, this becomes, in vector form,

$$\frac{1}{h}\nabla \cdot (h\nabla\eta) - \frac{h}{i\sigma}\nabla\eta \wedge \nabla\frac{f}{h} \cdot \hat{\mathbf{k}} - \frac{f^2}{gh}\eta = 0 + O\left(\frac{\sigma}{f}\right); \quad (1.2)$$

$\hat{\mathbf{k}}$ is a vertical (radial) unit vector. At a rigid vertical boundary with normal \mathbf{n} ,

$$\hat{\mathbf{k}} \wedge \nabla\eta \cdot \mathbf{n} = 0 + O\left(\frac{\sigma}{f}\right).$$

The limit $g \rightarrow \infty$ gives another possible equation, for then continuity is satisfied identically by a stream function for the mass flux,

$$\mathbf{k} \wedge \nabla \psi = h\mathbf{u}.$$

The equation is found by eliminating η from (1.1):

$$\nabla \cdot \left(\frac{1}{h} \nabla \psi \right) - \frac{1}{i\sigma} \left(\nabla \psi \wedge \nabla \frac{f}{h} \right) \cdot \mathbf{k} = 0. \tag{1.3}$$

(1.3) is rigorously correct with a rigid lid, and the inviscid condition at a vertical wall is simply $\psi = \text{constant}$. It has the advantage that the eigenvalues $\{\sigma_n\}$ corresponding to $\{\psi_n\}$ in a closed basin are real, while those corresponding to $\{\eta_n\}$, equation (1.2), are not. It is often desirable (in numerical computations, for example) that conservative equations remain so in spite of approximations.

Both (1.2) and (1.3) are vorticity equations. The added term in (1.2) represents the stretching of vortex lines by the free surface motion. For the surface to act rigid more is required than $\eta_t \ll \nabla \cdot (h\mathbf{u})$ in (1.1); flows may be dynamically divergent, yet kinematically non-divergent. In this case, if $f^2 L^2 / gH \ll 1$, the stretching term is negligible and the equations equivalent. They are not identical, but may be obtained from one another, to $O(\sigma/f)$, by substituting

$$\mathbf{k} \wedge \nabla \psi = \frac{gh}{f} \left(\mathbf{k} \wedge \nabla \eta - \frac{i\sigma}{f} \nabla \eta \right).$$

The implicit assumptions we have made are: (i) The non-linear terms are negligible. This requires that $\epsilon/\omega \ll 1$, where $\omega = \sigma/f_0$ and ϵ is based on *averaged* deep-ocean velocities. With ω small this is a more severe constraint than just geostrophy. It implies that fluid columns are not carried across the topography of interest during a period of the wave (see also page 183). (ii) The β -plane is valid. Topography tends to reduce the scale of the motions at a given frequency, and hence to improve this approximation. (iii) The horizontal component of $\mathbf{\Omega}$ is negligible. This is dictated by the large scale and presence of stratification, away from the equator. (iv) The stratification may otherwise be neglected. This is not always true, and will be discussed in a succeeding paper. (v) The horizontal velocities are independent of z . Rotation improves this traditional long-wave or ‘barotropic’ assumption. Consider, for example, the error in the horizontal vorticity equation:

$$-i\sigma(v_z - w_y) + fu_z = 0. \tag{1.4}$$

Equation (1.3) shows that w is unusually small in a quasigeostrophic balance, for $w/U \sim \omega H/L$, while long gravity waves have much more horizontal divergence $w/U \sim H/L$. Hence (1.4) shows that the non-dimensional shear of the horizontal velocity is

$$(H/U) u_z \sim \omega^2 (H/L)^2$$

if $v \sim u$. This is very small even when $H/L \sim 1$. (vi) Viscosity may be neglected. The application of simple quasi-steady Ekman theory, for small bottom slopes, implies that the motion decays exponentially, but without regard to horizontal scale. The spin-down time for the ocean, $H/(vf)^{\frac{1}{2}}$, would be about a month for barotropic motions, if the eddy viscosity were taken to be unity, but we do not

really know the nature of the turbulent lower boundary layer. The action of the tides (perhaps the largest velocities present at the bottom) in dissipating these motions has been considered by Groves (1965).

2. Depth variations in one direction

When the contours of depth are straight, the problem on a β -plane is separable at the lowest order in ω . Let us take the simplest case, when h is a function of latitude only. Then

$$\begin{aligned} \nabla^2 \psi - \frac{h_y}{h} \psi_y + \frac{i}{\omega} \left(\frac{1}{R} - \left(1 + \frac{y}{R} \right) \frac{h_y}{h} \right) \psi_x &= 0. \\ 1 \quad ML \quad \frac{1}{\omega} \left[\frac{L}{R} - \left(1 + \frac{L}{R} \right) ML \right], & \quad (2.1) \\ R = \frac{f_0}{\beta} = R_e \tan \Phi, & \end{aligned}$$

where Φ is the latitude at which $y = 0$. The relative magnitudes of the terms are indicated. The horizontal scale of ψ is L , that of the topography is a , and $h_y/h > \delta/a \equiv M$.

The second term may be neglected with error of order ML , $\sim \omega$, if the topography dominates (if the first and fourth terms balance), or $\sim \omega MR$ if the β -effect dominates (first and third terms). Thus depth variations are much more important in the horizontal divergence than in the vorticity term, reminiscent of the Boussinesq approximation. It also appears that the relative importance of topography and the β -effect is measured by MR .†

The variation of f (the fifth term) is $\lesssim L/R$ ($\lesssim \omega$). Hence with errors of order $\lesssim \omega$ (or, it turns out, $\lesssim \delta$), these two terms may be neglected. Veronis has pointed out to the author (see Veronis 1966) that a fallacious growth of the solutions with time may appear when the straight contours cross latitude lines, if f is held constant. The two approximations cited here, taken together, produce real frequencies for any bottom configuration, however, and give a consistent representation if the change in depth over a wavelength is small. Again, it is often desirable to deal with equations that remain conservative in spite of approximations.

For motions sinusoidal in x , (2.1) reduces to

$$\phi_{yy} + (E_1 - V_1(y)) \phi = 0, \quad (2.2)$$

where

$$\begin{aligned} \psi &= e^{ikx} h^{\frac{1}{2}} \phi, \\ E_1 &= -k^2 - \frac{k}{\omega R}, \\ V_1 &= -\frac{k}{\omega} \frac{h_y}{h} (1 + y/R) + \frac{1}{4} \left(\frac{h_y}{h} \right)^2 - \frac{1}{2} \left(\frac{h_y}{h} \right)_y. \end{aligned}$$

We now set $f = f_0$ (the second approximation) but retain $\nabla(1/h) \cdot \nabla \psi$ to demonstrate the accuracy of the first approximation.

† These arguments are somewhat limited, however. For long waves over 'roughness' ($L \gg a$) the parameter turns out to be MR (a/L) due to cancellation between positive and negative slopes.

Simple slope

When $h_y/h = M$, a constant or exponentially varying depth, the solutions are plane waves modulated in amplitude by $h^{\frac{1}{2}}$. The dispersion relation is

$$\omega = ([1/R] - M)k/[k^2 + l^2 + M^2/4], \tag{2.3}$$

where

$$\psi = h^{\frac{1}{2}} \exp [i(kx + ly)].$$

For the shorter waves the influence of the slope appears in parallel with $1/R$, which is thus modelled by a bottom shelving to the north. The contributions

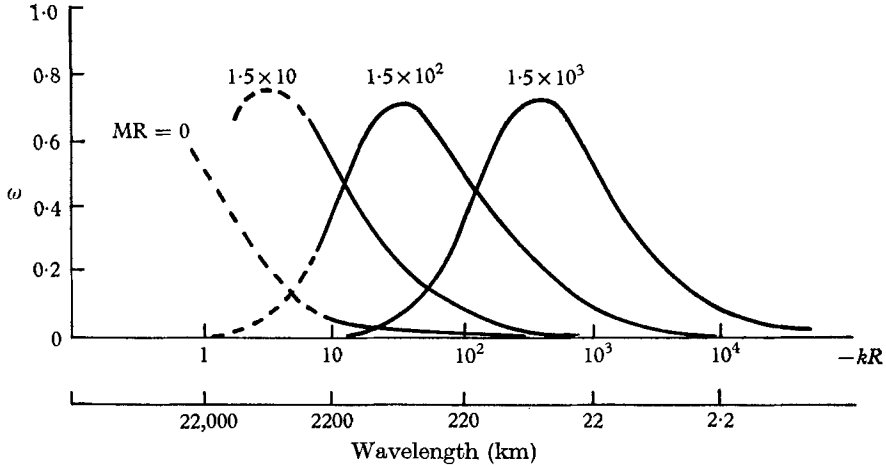


FIGURE 2. Dispersion relation for waves on a slope, $k = l$, for various MR ($M < 0$); $MR = 15$ corresponds to a slope of about 10^{-2} in the deep ocean. The curves are dashed where the approximation $f = f_0$ begins to fail. $R = 6.4 \times 10^8$ cm.

are equal, in the deep oceans at mid-latitudes, over a slope of only about 10^{-3} . Figure 2 shows how the frequency of a wave with crests inclined at 45° from east depends on wave-number and slope; the typical waves over a slope are much shorter than Rossby waves. The singular nature of the limit $\omega \rightarrow 0$ should be apparent from this solution. The streamlines need not become tangent to the f/h contours, for the scale of the derivatives can balance the smallness of ω .

The curves in figure 2 are dashed where the approximation $f \approx f_0$ begins to fail at mid-latitudes. For f to vary little over the wave scale we require $lR \gg 1$. At $\Phi = 45^\circ$ this implies that $l^{-1} \ll 7600$ km. Near the equator the situation is worse. At $\Phi = 10^\circ$, $l^{-1} \ll 1250$ km; that is, the wavelength must be much less than 5500 km. The solution fails, for example, in the gradually shelving Eastern Pacific Basin, where the slope is comparable to H/R over a vast region: variations in f will affect long waves there.

The highest frequencies occur at

$$k = \pm \frac{M}{2\sqrt{2}},$$

$$\omega = \frac{\sqrt{2}}{2} \left(1 - \frac{1}{MR} \right).$$

They describe waves for which the depth changes by a large factor over a wavelength. Since we shall usually be interested in the right-hand sides of the curves (the lower frequencies) we shall often neglect $M^2/4$ in the denominator of (2.3). This is equivalent to the first approximation mentioned above. In the diagram showing possible wave-numbers for a given frequency (figure 3) the approximation would move the circular locus tangent to the l -axis through a distance of order $\omega^2(MR)^2/(1+MR)$.

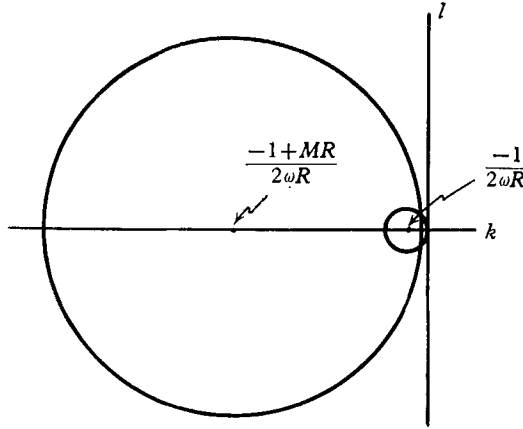


FIGURE 3. Locus of wave-numbers at a given frequency, depth contours east and west, plotted for $MR = -7.5$, $\omega = 0.1$. The small circle, for $M = 0$, describes Rossby waves. The large circle is slightly to the left of the l -axis.

When the straight contours of h do not run east and west, those of f/h are no longer straight and we make both approximations. Taking $h = h(\eta)$, where

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

are co-ordinates rotated from north and east through an angle α , the equation becomes

$$\hat{\psi}_{\eta\eta} + (E_2 - V_2(\eta)) \hat{\psi} = 0, \tag{2.4}$$

where

$$E_2 = \left(\frac{\sin \alpha}{2\omega R}\right)^2 - \frac{k \cos \alpha}{\omega R} - k^2,$$

$$V_2 = \frac{k h_\eta}{\omega h},$$

$$\psi = \hat{\psi}(\eta) \exp\{i(k\xi - (\sin \alpha/2\omega R)\eta)\}.$$

If the bottom slopes upward to the west, (2.4) with $h_\eta/h = M < 0$, $\alpha = \frac{1}{2}\pi$ gives for the dispersion relation

$$\left(l + \frac{1}{2\omega R}\right)^2 + \left(k - \frac{M}{2\omega}\right)^2 = \frac{R^{-2} + M^2}{4\omega^2},$$

with $\psi = \exp\{i(k\xi + l\eta)\}$ (ξ, k now refer to *north*). The frequency for the case $k < 0$, $l = \pm k$ (l is the total η wave-number) is plotted in figure 4, against l . The β -effect reinforces the wave with a component of its phase moving west,

raising its frequency, and detracts from the frequency of the eastward moving wave.

Since MR is the measure of topographic effects with a smooth slope, it appears that an ocean of scale L , with a bowl-shaped bottom, will be dominated by topography if only $L/R \ll 1$ (since $MR \sim R/L$). Many ocean basins of course have steep sides and level (but often rough) interiors: these are another problem.

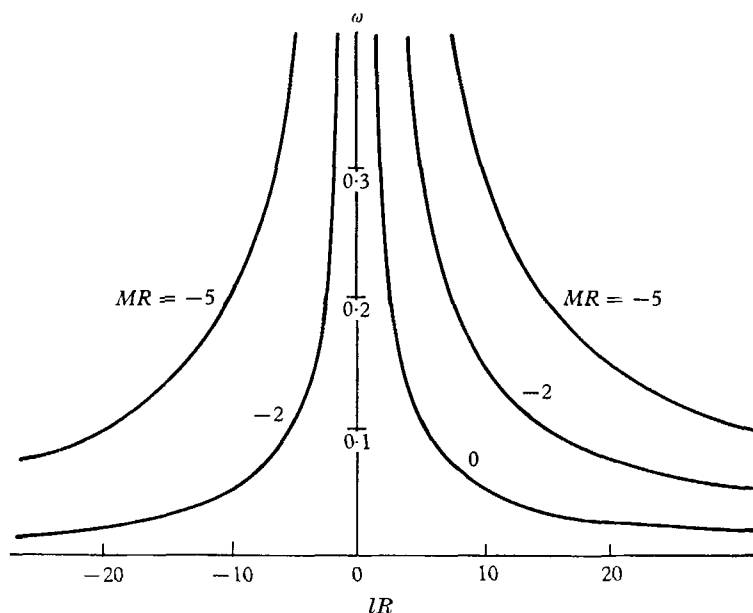


FIGURE 4. Dispersion relation for an upslope to the west; $k < 0$, $l = \pm |k|$. MR is the relative measure of the slope- and β -effects.

Abrupt topography; reflexion from a sloping step

The form of the one-dimensional equation suggests an analogy to quantum physics. If E_2 is the energy of, say, an electron passing through a steady potential field, $V_2(\eta)$, its wave representation is found from the Schrödinger equation (2.4). The frequency in our problem, however, appears in a more difficult position than does E_2 , so the eigenvalue problems do not correspond simply. When a plane Rossby wave incident from $\eta = -\infty$ comes upon a ridge of infinite extent in the ξ -direction, the slopes cause V_2 to be locally much greater than E_2 . ψ will then oscillate or decay rapidly in space, depending on the sign of $E_2 - V_2$, and this local behaviour presents a high impedance to the external field. Balancing this adverse effect on the passage of the wave, however, is the great wavelength in the exterior, which will tend to let it tunnel past.

If $h(\eta)$ has the form of a rounded step the analogous potential has the even form of a well or barrier. When $V_2 \propto \tanh^2 \eta$ [$h \propto \exp(\tanh \eta)$] (2.4) may be reduced to a Legendre equation (see Morse & Feshback 1953). The detailed solution in hypergeometric functions is known (by a generation of physicists) to justify the simpler approach of matching solutions with broken-line profiles of V_2 , for long waves. We thus find that the normal and tangential velocities,

proportional to $\hat{\psi}$ and $\hat{\psi}_r$, are quantities to be held continuous at a discontinuity in *slope*, as we should expect.

Consider a wave incident from the south on a step running east and west, with its face sloping between $0 \leq y \leq a$ (as in figure 5). Taking account of the radiation condition beyond the step, the form of the solution is

$$\begin{aligned} \hat{\psi} &= A_1 e^{i\mu y} + A'_1 e^{-i\mu y} & (y \leq a), \\ &= A_2 e^{\mu y} + A'_2 e^{-\mu y} & (0 < y \leq a), \\ &= A_3 e^{i\mu y} & (y > a), \end{aligned}$$



$$h/H = 1 \qquad = e^{\delta(y/a)} \qquad = e^\delta$$

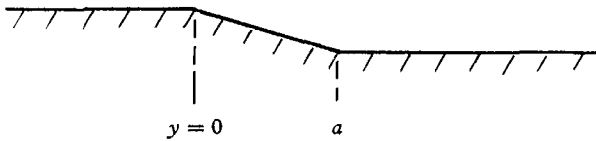


FIGURE 5. Form of the sloping step.

where

$$\begin{aligned} \bar{\psi} &= \hat{\psi}(y) \exp\{i(kx - \omega f_0 t)\}, \\ l^2 &= \frac{-k}{\omega R} - k^2 = E, \\ \mu^2 &= -l^2 - \frac{k}{\omega} M = V - E, \\ M &= \frac{h_y}{h} \equiv \frac{\delta}{a}. \end{aligned}$$

$R(\bar{\psi})$ is the total stream function and δ the net fractional change in depth. The group velocity of the incident wave must be directed towards the step, that of the transmitted and reflected waves away from the step, and therefore l is negative.

The matching of velocities at $y = 0, a$ gives

$$\begin{aligned} A_1 + A'_1 &= A_2 + A'_2, \\ A_2 e^{\mu a} + A'_2 e^{-\mu a} &= A_3 e^{i\mu a}, \\ i l (A_1 - A'_1) &= \mu (A_2 - A'_2), \\ \mu (A_2 e^{\mu a} - A'_2 e^{-\mu a}) &= i l A_3 e^{i\mu a}. \end{aligned}$$

It follows that the fraction of the incident energy transmitted is, to order δ ,

$$\begin{aligned} T_1 &= \left| \frac{A_3}{A_1} \right|^2 \\ &= \left[1 + \frac{1}{4} \left(\frac{\mu}{l} + \frac{l}{\mu} \right)^2 \sinh^2 \mu a \right]^{-1}, \end{aligned} \tag{2.5}$$

which is plotted in figure 6, for $M > 0$, $l = k$ (the step falls to the north). For a given incident wave, at 45° to the slope, say, and fractional change in depth, δ , T_1 varies little with the width of the step until $(\omega/\delta)ka \sim 1$.

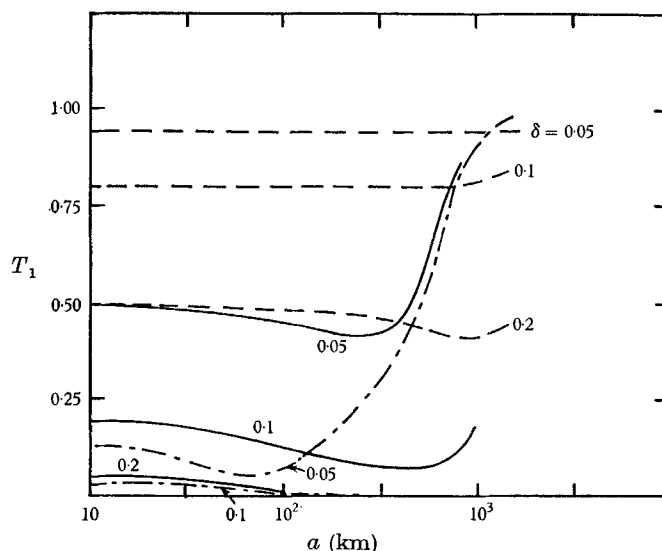


FIGURE 6. Transmission across a sloping step, as a function of its width; $h = h(y)$, $M > 0$, $k = l$, $\Phi = 45^\circ$. ---, $\omega = 0.1$, $T = 7.1$ days (wavelength 5700 km); —, $\omega = 0.025$, $T = 28$ days (wavelength 1400 km); -.-, $\omega = 0.01$, $T = 71$ days (wavelength 570 km).

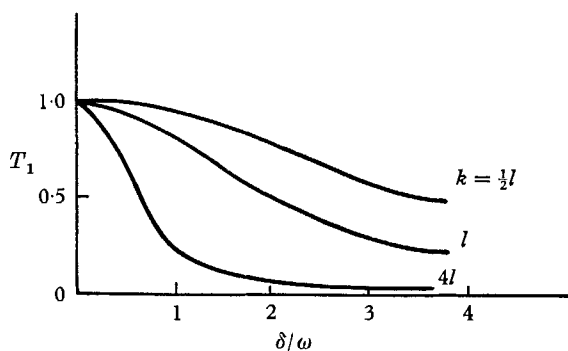


FIGURE 7. Effect of direction of incident wave on transmission across an abrupt step.

If $M < 0$ the step rises to the north and the potential 'barrier' becomes a 'well'; in this region the eigenfunction always curves towards the zero axis. With $\mu \rightarrow i l_2$ in (2.5) T_1 oscillates as a function of the step width, a , between the values $[1 + \frac{1}{4}([\mu/l] + [l/\mu])^2]^{-1}$ and one. This latter limit, representing perfect transmission, occurs when an integral number of half-waves fit in over the slope, so that no reflected wave is needed in the matching. In physics the analogue for centred scattering is called the Ramschauer effect. Near these wavelengths there are occasional resonances in which the long-slope velocities are much larger than any exterior velocity.

Abrupt step

Since Rossby waves are so long, the limiting value for $\mu a \rightarrow 0$ will usually be applicable. For either well or barrier this is

$$T_1 = \left[1 + \frac{1}{4} \left(\frac{\delta k}{\omega l} \right)^2 \right]^{-1} \quad (2.6)$$

for $(\omega/\delta)ka \ll 1$.

If the step lies at an angle α with respect to east, k and l refer to (ξ, η) coordinates and

$$l^2 = \left(\frac{\sin \alpha}{2\omega R} \right)^2 - \frac{k \cos \alpha}{\omega R} - k^2. \quad (2.7)$$

Relation (2.6) is plotted in figure 7 to show the effect of the direction of the incident wave; the step is a directional filter for Rossby waves, passing only those with crests at an angle of less than $\tan^{-1}(\delta/\omega)$ from the depth contours. Clearly, a wave with velocities paralleling the contours will have no horizontal divergence forced upon it and suffer no reflexion.

In the limit of long waves T_1 may be found directly by integrating (2.4) across the step to give a matching condition for the long-slope velocity, u . For a small step at $\eta = 0$ this is

$$\begin{aligned} [u]_{0-}^{0+} &= [\psi_\eta]_{0-}^{0+} \\ &= \frac{k}{\omega} [h]_{0-}^{0+} \psi(0) \\ &= \frac{\delta}{\omega} v(0), \end{aligned}$$

where the upslope velocity, v , is continuous. The change in u could take place over 100 km and yet look to a long Rossby wave like a sheet of vorticity. For a typical wave with $v \sim u$ in the exterior, large δ/ω forces $v(0)$ to be almost zero; the step then acts like a rigid wall. The time-dependent vortex sheet could not, by Kelvin's theorem, exist in a flat-bottomed ocean, but here the change in depth releases relative vorticity in order to conserve potential vorticity. The explanation, in terms of momentum, is that fluid forced up the slope alters its velocity, by continuity, and hence feels a stronger Coriolis force deflecting it to the right.

This argument leads us to expect that, even when f changes substantially over a wavelength, or when the step is not uniform, there will still be a barrier if δ/ω is everywhere greater than one.

This limit of an abrupt step gives us another result of interest, for we may find the effect of a step of *arbitrary* height. The full equation may be integrated across $\eta = 0$ since ψ changes much more slowly than h . This gives

$$[u]_{0-}^{0+} = \left[\frac{\psi_\eta}{h} \right]_{0-}^{0+} = \frac{k\psi(0)}{\omega} \left[\frac{1}{h} \right]_{0-}^{0+}$$

for the strength of the vortex sheet. With, also,

$$[\psi]_{0-}^{0+} = 0$$

the matching equations determine the energy transmission:

$$T_2 = 4D \left/ (D+1)^2 + \left(\frac{kD-1}{l} \frac{D-1}{\omega} \right)^2 \right.$$

where $D = h(0-)/h(0+)$ and f is still treated as a constant.

As expected from (2.6), unless k/l is small, virtually no energy can penetrate into a region for which $\delta (\equiv D-1)$ is of order unity. In view of our interest in finding Rossby waves in surface-elevation measurements over the continental shelf, we should keep in mind a result for long gravity waves in the same situation; as $\delta \rightarrow 1$ the transmitted *amplitude* over the incident amplitude goes to 2.0 though the energy is almost totally reflected; the transmitted wave is measurable. The ratio of surface heights for Rossby waves, however, is

$$\left(\frac{T_2}{D} \right)^{\frac{1}{2}} \rightarrow \frac{2\omega l}{D k}$$

as $D \rightarrow \infty$. The attenuation will thus make observations at a coast a poor representation of abyssal motions. When the shelf inside of the rim rises moderately to the coast (say $h \propto \eta$) the surface motion does not vanish as it would next to a vertical wall but reaches a finite maximum. Those waves that do manage to penetrate the rim might thus be measured by tide-gauges at the coast.

Regions of different mean depth should, in general, begin to be isolated from each other when $\delta/\omega \sim 1$. The North Atlantic at the latitude of the British Isles, for instance, will not feel Rossby waves on the deeper plains to the south.

Reflexion from a ridge

Since the lengths involved are great it is natural to wonder how other, nearby changes in depth will affect this result. The step is, in fact, an atypical example of an isolated feature because it is really the line of separation between regions of different quiescent potential vorticity.

We therefore consider waves incident on a ridge of infinite length that rises out of a flat plain. The corresponding V now represents a well followed (or preceded) by a barrier. The integral of this potential vanishes and so the same arguments that predict reflexion by a narrow step, alone, show that the combined effects cancel, leaving the narrowest ridges transparent; it is the net change of depth that counts.

Consider the abrupt ridge of finite width given by

$$\begin{aligned} h &= H & (\eta > a, \quad \eta < 0) \\ &= H(1-\delta) & (0 \leq \eta \leq a). \end{aligned}$$

We expect physically that the reflexion of a long wave will require the strength of the 'net' vortex sheet,

$$S \equiv \frac{[u]_{a-}^{a+} - [u]_{0-}^{0+}}{U},$$

to approach unity, where the typical external velocity is U . Each individual sheet is driven by the local upslope velocity, v , so

$$S = \frac{(\delta/\omega)[v(a) - v(0)]}{U}.$$

The separation must be great enough for the v 's to differ.

If the reflexion coefficient, R , is assumed small, but the sheets individually strong ($\delta/\omega \gg 1$) and the ridge narrow ($ka \ll 1$), then

$$\begin{aligned} |v(a) - v(0)| &\approx av_{2,\eta} \\ &= au_{2,\xi} \\ &= kau_2, \end{aligned}$$

where v_2, u_2 are typical velocities over the ridge, and with variation $\propto e^{ik\xi}$ along the ridge.

$$\begin{aligned} \text{Now} \quad u_2 &\approx [u]_{0-}^{0+} \\ &= \frac{\delta}{\omega} v(0), \end{aligned}$$

because $[u]_{0-}^{0+} \gg U$ assuming the reflexion weak (such intense velocities did not appear over a step, except in the unusual resonances of the 'well' profiles). The change in v , which is also proportional to the change in $\hat{\psi}$, is thus

$$v(a) - v(0) \approx ka(\delta/\omega)v(0).$$

It follows that the net strength is

$$\begin{aligned} S &= \left(\frac{\delta}{\omega}\right)^2 ka \frac{v(0)}{U} \\ &\sim \left(\frac{\delta}{\omega}\right)^2 ka, \end{aligned} \tag{2.8}$$

for an obliquely incident wave, and it is inconsistent to predict $R \ll 1$ if this approaches unity. Although the reflexion is thus greatly reduced, δ/ω is often large enough for it to be significant.

By matching solutions for the abrupt ridge we find, in fact,

$$\begin{aligned} T_3 \equiv \left|\frac{A_3}{A_1}\right|^2 &= [1 + 2\gamma(\gamma + 1)(1 - \cos 2la)]^{-1}, \\ \gamma &= \frac{1}{4} \left(\frac{\delta k}{\omega l}\right)^2, \end{aligned}$$

which indeed depends on the above parameter S for $ka \ll 1$.

A ridge of finite length

The derivations above require that the topography extend many wavelengths along its contours without change, but the vortex sheets will still appear when a ridge is terminated at a length, D , many times its width, even if both dimensions are much less than a wavelength. If, in the argument leading to (2.8), we say that

$$u_{2,\xi} \sim u_2/L,$$

where L lies between D and the wave-scale, k^{-1} , we have an estimate of the 'net' vortex sheet:

$$S \approx \left(\frac{\delta}{\omega}\right)^2 \frac{a}{L}.$$

This shows the surprising tendency for the response to be *stronger* if the ridge is of finite length. The topography will begin to scatter waves as strongly as the vertical walled island formed by its circumscribing cylinder when S approaches one.

The change in ψ or pressure between front and back will cause a periodic vortex-like secondary flow at the ends of the ridge. Similar, but steady, regions of vorticity appear at the tips of aeroplane wings due to the pressure drop in the primary flow between top and bottom.

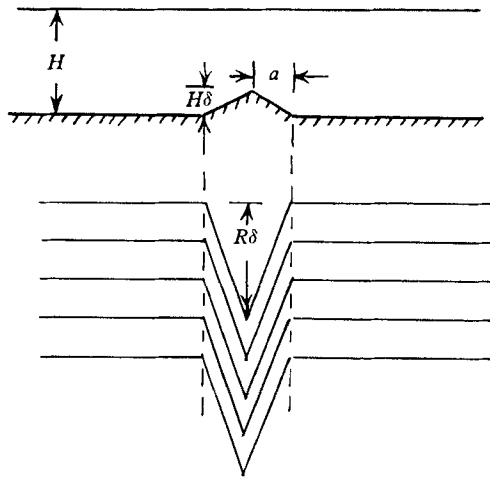


FIGURE 8. f/h contours for a north-south ridge with triangular cross-section. The maximum southward displacement of a contour is $R\delta$.

Triangular ridge

Actual topography is usually less abrupt than the models used above. We therefore solve, again by matching velocities at discontinuities in slope, for the transmission past an infinite ridge with a smoother, triangular cross-section and small height:

$$\begin{aligned} h_\eta/h &= -\delta/a \quad (-\frac{1}{2}a < \eta \leq 0), \\ &= +\delta/a \quad (0 < \eta \leq \frac{1}{2}a) \end{aligned}$$

(the f/h contours for this model are shown in figure 8). This gives six simultaneous equations for the six unknown amplitudes of waves in the four regions.

The result for the transmitted energy is

$$\begin{aligned} T_4 = & \left\{ \left[\cosh(\mu a) \cos(\alpha a) - \frac{1}{2} \left(\frac{\alpha}{\mu} - \frac{\mu}{\alpha} \right) \sinh(\mu a) \sin(\alpha a) \right]^2 \right. \\ & \left. + \frac{1}{4} \left[\left(\frac{\alpha}{l} + \frac{l}{\alpha} \right) \cosh(\mu a) \sin(\alpha a) + \left(\frac{l}{\mu} - \frac{\mu}{l} \right) \sinh(\mu a) \cos(\alpha a) \right]^2 \right\}^{-1}, \end{aligned}$$

where l , α and $i\mu$ are the η wave-numbers of the exterior, and the two interior regions, respectively:

$$l^2 = -k^2 - k \cos \alpha / \omega R + (\sin \alpha / 2\omega R)^2,$$

$$\alpha^2 = l^2 + \delta k / \omega a,$$

$$\mu^2 = -l^2 + \delta k / \omega a.$$

When $\mu^2 < 0$ (wave-like solutions in all four regions) T_4 holds with $\mu = i|\mu|$.

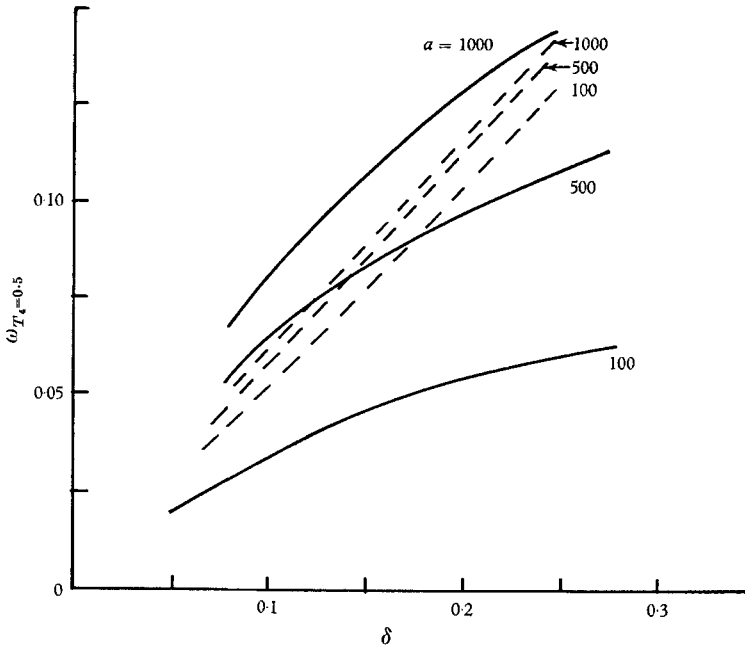


FIGURE 9. Transmission past a ridge: the frequency of marginally reflected waves (for which $T_4 = 0.5$) is plotted against the ridge height for several half-widths, a (solid curves). The same relation is also shown for sloping steps of the same characteristic height, and of width a (dashed curves).

T_4 decreases monotonically with increasing half-width, a , or increasing δ , within the interesting range of values. The results are summarized for a ridge running north and south in figure 9, where the critical frequency at which $T_4 = 0.5$ is plotted against δ for several a . The incident wave is always taken with its group velocity at 45° to the ridge ($k = l$), which causes neither the strongest nor the weakest reflexion. f corresponds to 35° north latitude. T_4 generally becomes quite small when a good fraction of a wave (over the slope) is contained within a distance a : higher-frequency (longer) waves are transmitted. The earlier 'narrow-ridge' estimate (2.8) applies here only when $(\delta/\omega)ka \ll 1$ (T_4 then depends upon the same parameter). The critical frequency it gives varies like $\delta^{\frac{2}{3}}(a/R)^{\frac{1}{3}}$, which in fact describes the whole range of figure 9 surprisingly well.

For $k = l$ the wavelength corresponding to the critical ω can take on two values, depending on the sign of the westward component of group velocity. The

ridge is thus a greater barrier to waves of given scale generated to its east, than to its west.

It is interesting to compare the results, plotted with dashes, for sloping steps corresponding to the first half of the ridge. These are close to the 'abrupt-step' value, $\omega_c = \delta/(2 + \delta)$. With $a = 100$ km, the steps are the more effective barriers, but with wider profiles the ridge is more effective for small δ , and so the cancellation of the vortex sheets is not always serious. There is, however, a clear tendency for higher ridges to be less obstruction than the corresponding steps (the dashed lines may be extended almost straight as δ increases but the ridge derivation is not valid for δ of order unity).

Applications

The earth's surface has evolved on a scale so large that these results should find application in spite of the one-dimensional profiles used; the Mendocino Escarpment in the Pacific is a good example of straight contours.

Perhaps the most prominent topographic feature to be found in the oceans is the Mid-Atlantic Ridge and its continuations that snake over 40,000 miles of the surface of the earth. These calculations show that it is massive enough to affect most of the long Rossby waves in the Atlantic. The ridge ranges in fractional height from 1 at the Azores to about 0.35 at 15° N. Smaller ridges, for which these results are valid (say $\delta = 0.15$, $a = 750$ km, 35° N), reflect a majority of the energy in waves for which $\omega \lesssim 0.1$; that is, wavelengths $\lesssim 7000$ km (group velocity to the west), or $\lesssim 3000$ km (group velocity to the east). The decrease in f with latitude should be at least compensated by the greater δ , down to about 15° N. The basins on either side of the ridge should thus support their own normal modes, without communication between them. The fundamental mode in the Atlantic, however, should be unaffected by the lower parts of the ridge, although the region of the Azores will distort it somewhat. This effect more resembles centred scattering, which will be treated below.

Pedlosky (1965) gives an interesting comment on the importance of Rossby waves, which is relevant here. In an ocean of constant depth, energy is carried efficiently to the west by long waves. They are reflected, however, as short waves (whose group velocity is to the east), and these tend to be damped by friction before reaching mid-ocean. The western edge of the ocean thus tends to be a sink of quasigeostrophic energy, and should be the site of much time-dependent eddying (see also Phillips 1966). The frictional Gulf Stream is the result of the low-frequency limit of this argument.

For the same reason the eastern edge of the mid-Atlantic Ridge should support strong eddying, in regions where the adjacent abyssal plain is smooth enough for Rossby waves to propagate.

3. Wave trapping

In figure 1 there are regions, dense with contours, that are coherent over long distances due, for instance, to the continental rises and Mid-Atlantic Ridge. The earlier demonstration that a sloping step sometimes acts as a potential well

suggests that, as well as reflecting external waves, such profiles might support internal reflexions, preventing forced motions from escaping the region of the slope.

Abrupt step

Over a sloping step a whole family of oscillations exists with frequencies that decrease as the change in phase between the edges of the step increases. For simplicity we first limit our attention to the ‘lowest’ mode, a wave much longer than the width of the step, a , which has no nodes parallel to the contours. In this limit the slope may be considered impulsive, lying along $\eta = 0$. First setting $\beta = 0$, but including the term formerly neglected in the vorticity, $\nabla(1/h) \cdot \nabla\psi$, the equation is

$$\left. \begin{aligned} h \left(\frac{1}{h} \hat{\psi}_\eta \right)_\eta + \left(-k^2 + \frac{k}{\omega} \frac{h_\eta}{h} \right) \hat{\psi} &= 0; \\ \bar{\psi} = \hat{\psi}(\eta) \exp \{i(k\xi - \omega f_0 t)\}, \quad \frac{h_\eta}{h} &= \Delta(\eta) \delta, \end{aligned} \right\} \quad (3.1)$$

where $\Delta(\eta)$ is the delta function. The solutions are tent-shaped,

$$\hat{\psi} = e^{k|\eta|} \quad (k < 0),$$

for

$$\nabla^2 \hat{\psi} = 0$$

where $y \neq 0$, and continuity requires that

$$\hat{\psi}(0+) = \hat{\psi}(0-).$$

The strength of the vortex sheet is, again

$$\left[\frac{1}{h} \hat{\psi}_\eta \right]_{0-}^{0+} = \frac{k}{\omega} \hat{\psi}(0) \left[\frac{1}{h} \right]_{0-}^{0+}.$$

We find, for an arbitrarily high step,

$$\omega = \frac{\delta}{2 + \delta} \operatorname{sgn}(k),$$

where

$$\delta \equiv \frac{h(0+) - h(0-)}{h(0-)};$$

$\omega > 0$ by definition, and so the phase always moves with deep water to the left.

The frequency is thus independent of the scale of the motion, and of the form of its ξ dependence; in this limit energy cannot be guided along the step. Since the scale k^{-1} can be as small as the depth of the ocean, the frequency should reveal the local topographic height. When δ is at all substantial the frequency will be as high as that of the longest Rossby waves.

The surface has been taken to be rigid, so that gravity is irrelevant to the solution (this is not a Kelvin wave). Free surface motion is in fact of no dynamical importance if f^2/gHk^2 is small. With $H = 3$ km this requires only that the wavelength be much less than 11,000 km, which it will be in most cases of interest.

Inclusion of the β -effect

The strong topographic restoring force is embedded in the weaker, but extensive, field due to β . As the scale k^{-1} increases to that of Rossby waves of the same frequency, $\beta \equiv f/R$ must eventually become important. We choose waves that vanish at $\eta = \pm \infty$, eliminating the Rossby waves already treated:

$$\psi = \exp(-\lambda |\eta|) \exp(i[k\xi + l\eta]).$$

The matching conditions across a step aligned in the ξ -direction, at an angle α from east ($-\frac{1}{2}\pi < \alpha \leq \frac{1}{2}\pi$), are as above. The equation for the exterior, $\eta \neq 0$, gives

$$l = \frac{\sin \alpha}{2\omega R},$$

$$\lambda^2 = k^2 - \left(\frac{\sin \alpha}{2\omega R}\right)^2 + \frac{k \cos \alpha}{\omega R}.$$

With the matching condition

$$\lambda(\delta + 2) = -(k/\omega)\delta, \tag{3.2}$$

we have

$$\omega = -\frac{\cos \alpha}{2kR} + \sqrt{\left[\frac{1}{(2kR)^2} + \left(\frac{\delta}{2+\delta}\right)^2\right]}, \tag{3.3}$$

which is plotted in figure 10. The sign of the radical has been chosen to make $\lambda > 0$. With the convention $\omega > 0$, always, k has the same sign as δ , and the phase again moves with deep water to the left.

To understand the interaction of the two effects we consider the possible orientations of Rossby waves with frequency $\delta/(2+\delta)$. The locus of wave-numbers is, of course, a circle just to the left of the l axis in (k, l) space.

Letting the length scale of a purely topographic wave increase, we see that the first Rossby waves encountered (the shortest) are directed due west and expect, for $\delta < 0$, that this will reinforce the westward component of phase velocity in topographic motion, raising the frequency. For $\delta > 0$ the purely topographic wave has phase moving with an eastwards component. The competition with the Rossby wave of the same scale lowers the frequency at first. As the scale increases, however, with frequency still near $\delta/(2+\delta)$, the Rossby wave-number swings about its circular locus until eventually it has a component in the *other* direction along the ridge. The two effects can then combine, and the frequency increases. The exceptional cases are $\alpha = \frac{1}{2}\pi$, where the phase moves north-south, and the curves for $\delta < 0$ and $\delta > 0$ are symmetrical, and $\alpha = 0$, $\delta > 0$ where projection of the Rossby wave's phase can never move eastward along the step, and the competition continues to the end.

The form of the waves is shown in figure 11. The nodal lines move obliquely to the step in general. As $k \rightarrow 0$ the tent-shaped envelope of $\hat{\psi}$ raises itself and becomes flat, and the vortex sheet becomes insignificant. The limiting Rossby wave has

$$\frac{l}{k} = \frac{\sin \alpha}{\pm 1 - \cos \alpha}$$

$$= -\tan \frac{1}{2}\alpha \quad (\delta > 0),$$

$$= \cot \frac{1}{2}\alpha \quad (\delta < 0).$$

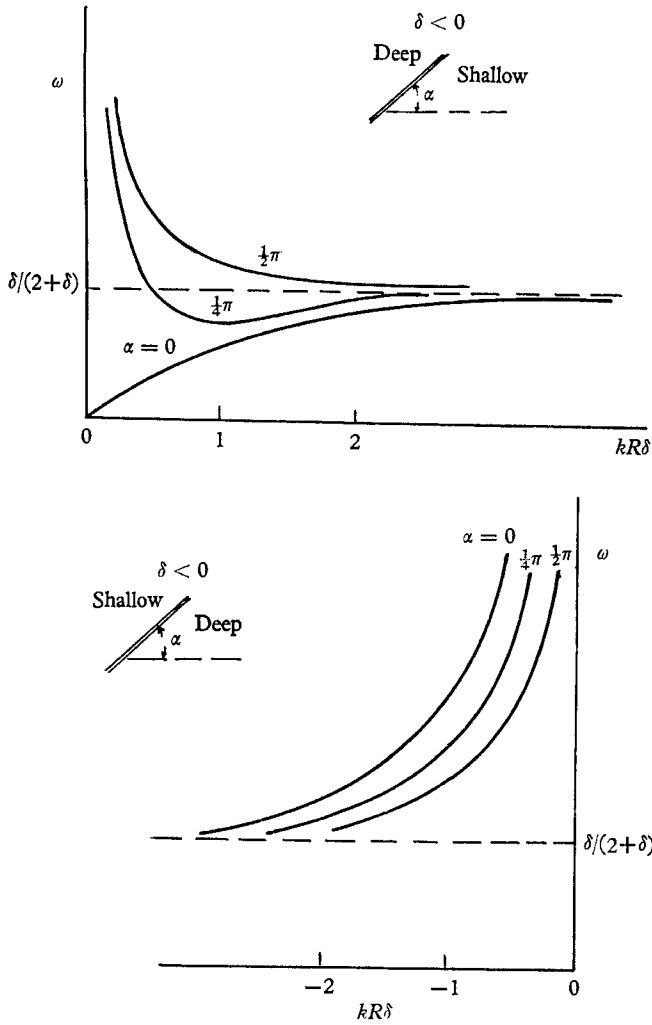


FIGURE 10. Effect of $\beta \equiv f/R$ on the dispersion of trapped waves. For $\delta \leq 0.1$ only those longer than $\sim 6 \times 10^4$ km are affected. k is the wave-number along the step.

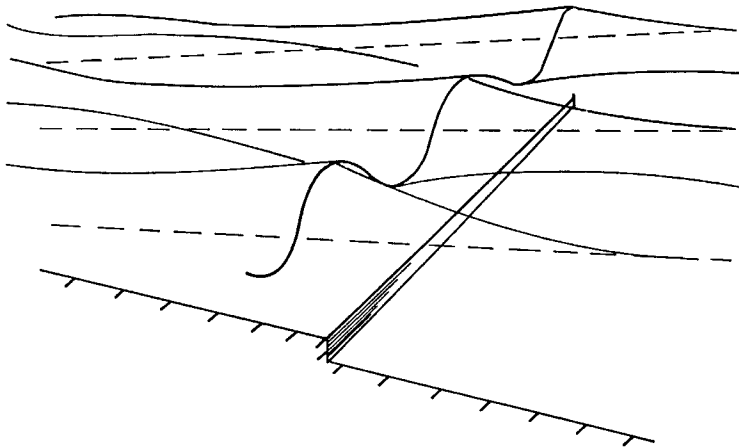


FIGURE 11. Form of ψ and the free surface for the oscillations due to a step in the ocean floor.

These are the two possible waves with group velocities along the step, in the same sense as the phase of the respective trapped modes.

The transition between types occurs at

$$k = \cos \alpha / 2\delta R.$$

If $\delta = 0.1$ this is at $k \simeq 10^{-8}$,

or a wavelength of greater than 6000 km. β should not, therefore, affect the trapping in most cases, except in providing a small group velocity along the step.

Higher modes over a step of finite width

There is a family of waves, of smaller scale than the fundamental, over the sloping step in figure 5. With $\delta \ll 1$ and β neglected for simplicity (implying $\omega k R \gg 1$ where k is again the long slope wave-number), the matching of velocities at $\eta = 0$, a gives the following relations:

$$\begin{aligned} l \tan \frac{1}{2} la &= \lambda \quad \text{for } \hat{\psi} \text{ even about } \frac{1}{2} a, \\ l \cot \frac{1}{2} la &= -\lambda \quad \text{for } \hat{\psi} \text{ odd about } \frac{1}{2} a, \end{aligned}$$

where

$$l^2 = -k^2 + k\delta/a\omega, \quad \lambda^2 = k^2$$

give the wave-number up the slope, and the decay constant outside the slope, respectively.

These equations are of the form

$$q(\omega, k) \begin{Bmatrix} \tan \\ \cot \end{Bmatrix} (q(\omega, k)) = \{\pm\} r(k),$$

which are tabulated for $q(r)$. The dispersion curves that result are given in figure 12. The long wave limit leaves, as the only unsteady motion, the mode already calculated, $\omega = \frac{1}{2}\delta$ ($\delta \ll 1$). The figure shows that this mode has its group velocity always in the opposite sense to the phase velocity. For $ka \ll 1$

$$\omega = \frac{\delta}{2 + |k|a} \operatorname{sgn}(k),$$

$$f_0 \omega_k = -\frac{1}{4} f_0 \delta a \quad (= -\frac{1}{2} ka \times (\text{phase velocity})),$$

independent of k . If a disturbance excites a set of such waves, all much longer than a , their form will move slowly and without dispersion along the step. Energy is carried a distance $\pi\delta a$ in a day, which is not far unless δ is substantial. The sense of the transport is to the right, for an observer facing shallow water.

The frequencies of the higher modes go to zero with ka , but they are still important. With $ka \rightarrow 0$, $\hat{\psi}_\eta \rightarrow 0$ at the edges of the step, $\eta = \pm \frac{1}{2} a$, and the eigenvalues are simply

$$\begin{aligned} \omega &= \frac{\delta ka}{2\pi} (1, \frac{1}{2}, \dots) \quad (\hat{\psi} \text{ even}), \\ &= \frac{\delta ka}{\pi} (1, \frac{1}{3}, \dots) \quad (\hat{\psi} \text{ odd}). \end{aligned}$$

The topography is then a non-dispersive wave guide. The group velocity may exceed that of the fundamental mode (and is in the opposite direction):

$$f_0 \omega_k = \frac{f_0 \delta a}{2\pi} (1, \frac{1}{2}, \dots) \quad (\hat{\psi} \text{ even}),$$

$$= \frac{f_0 \delta a}{\pi} (1, \frac{1}{3}, \dots) \quad (\hat{\psi} \text{ odd}).$$

With $a = 100 \text{ km}$ and $\delta = 0.2$, $c \leq 64 \text{ cm/sec}$ and hence even the lower frequency modes may be efficient carriers of energy, in the face of non-linearities.

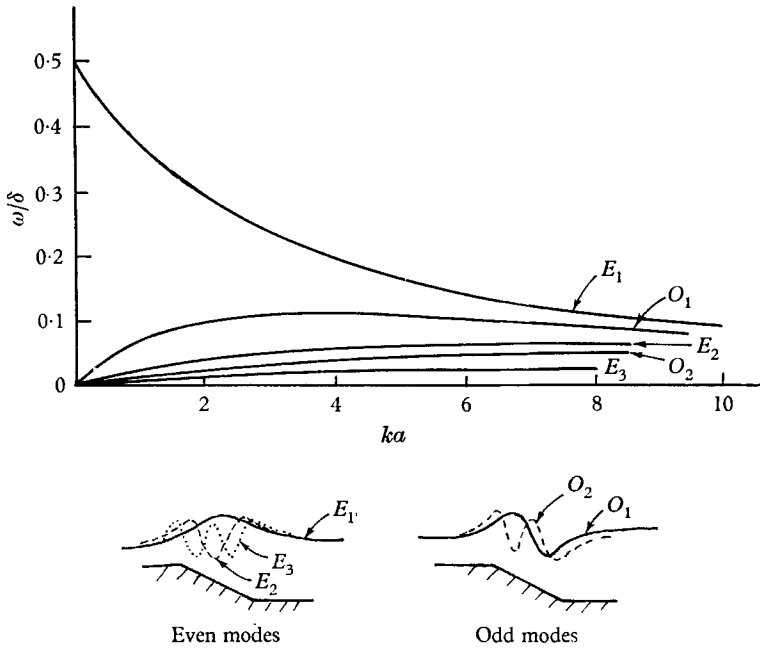


FIGURE 12. Dispersion relation and form of ψ for oscillations over a sloping step.

The low frequency limit supplies the motions that would occur in the wake of an obstacle over a slope or sloping step, with a slow current. As $\omega \rightarrow 0$ energy moves along the contours to the left (looking up the slope), drawing out the wake in this direction. In the same way, a slowly moving forcing effect will draw or push water along the contours to the left, but not to the right, as Lighthill (1967) showed for the predominantly westward influence that occurs in a constant depth ocean.

If a step is terminated laterally (perpendicular to the depth contours) by walls, or is cut off, an argument similar to that on page 177 indicates that energy can pile up and be dissipated at the *left* end (facing shallow water, in the northern hemisphere). One might, for example, find especially vigorous eddying at terminations along continental rises or the Mendocino Escarpment.

Longuet-Higgins (1968*a, b*) has extended these results to include free-surface divergence. The effect is rather unimportant unless the wavelength approaches

11,000 km, say for $H = 3$ km. The frequency of longer waves is then lowered since the raising of the free surface due to an upslope motion opposes spin-down. The modifications due to β often will occur before divergence is important.

Trapping over a ridge

Waves trapped over a ridge, with the same depth at either side, may be quite different. Thinking of the profile as two opposing steps, we see that the phase tends to move in opposite directions on the two sides, and expect some cancellation. With $R = \infty$ again, for simplicity, we set the incident wave to zero in the equations that dealt with the 'abrupt' ridge:

$$\begin{aligned} h &= H(1 - \delta) & (0 \leq \eta \leq a) \\ &= H & (\text{elsewhere}) \end{aligned}$$

and solve for the dispersion relation implied by the homogeneity of the matching equations. This gives

$$\begin{aligned} \omega &= \frac{1}{2}\delta(1 - \exp[-2|k|a])^{\frac{1}{2}}, \\ \text{or} \quad \omega &\approx \delta(\frac{1}{2}|k|a)^{\frac{1}{2}} \quad \text{for} \quad |k|a \ll 1. \end{aligned}$$

The waves may move in either direction along the ridge, but their form is lopsided in favour of the side over which the phase moves with deep water to the left. The frequency is lower than in earlier examples, but k can easily be large enough to 'separate' the two sides. The group velocity along the ridge is

$$f\omega_k = \pm \frac{1}{2}f\delta a \exp[-2|k|a](1 - \exp[-2|k|a])^{-\frac{1}{2}},$$

in the same sense as the phase velocity. A narrow ridge is dispersive, the group velocity approaching one-half the phase velocity for small ka . This calculation shows the nature of the long wave; a large ridge with sloping faces will support higher modes, as in the preceding section.

Non-linear behaviour

The most vulnerable assumption made so far is that the spin-up of a column of fluid is well represented by a balance at a point fixed in space. The measure of non-linearities is usually taken to be the Rossby number, $\epsilon \equiv U/fL$, where U and L are typical velocity- and length-scales of the motion, respectively, but this applies only to the momentum equation, which is dominated by the geostrophic balance whether the vorticity equation is linear or not. We here require ϵ/ω , a measure of the wave steepness (horizontal displacement \div wavelength) to be small. Since ω is the fundamental small parameter for the problem, there is a class of quasi-steady geostrophic motions which overlaps with our interest.

Scale analysis is of little help, since we have little prior knowledge of the appropriate L . For isolated profiles treated so far, the preservation of frequency meant that fluid was strongly deflected and the advection of vorticity reduced. For a step, in fact, the upslope velocity is smaller than that in the exterior by a factor of order $T_1^{\frac{1}{2}}$, although the length scale is also reduced. The results imply

$$\begin{aligned} \frac{\epsilon}{\omega} &\sim \frac{U(0)l_2}{f_0\omega} \\ &\sim \frac{U\sqrt{ka}}{f_0\omega a}, \end{aligned}$$

which is somewhat better than at first sight; if $\omega = 10^{-1}$, $k = 10^{-8} \text{ cm}^{-1}$, $a = 10^6 \text{ cm}$

$$\frac{\epsilon}{\omega} \sim 10^{-2} U \text{ (cm/sec),}$$

which should often be small.

When we come to treat continuously varying topography, this contrast in lengths is not as strong, and fluid will cross contours at large angles, raising ϵ/ω . There is, however, comfort to be found in the basic vorticity theorems. Suppose,

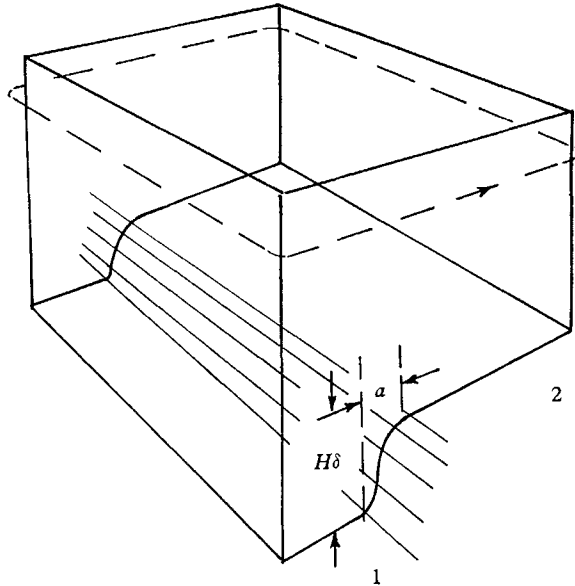


FIGURE 13. Contour for determining non-linear reflexion by a step.

for example, the motion takes place over roughness of small scale, L_R , in the sea floor. If $U/\omega f L_R \gg 1$ the flow is locally irrotational (to the extent that the fluid is inviscid and homogeneous), and the streamlines are compressed only near the bottom. If $U/f L_R \ll 1$, $U/\omega f L_R \gg 1$ the field is two-dimensional, with some advection of vorticity, depending on the extent to which the steady flow may move along contours. The effect of vortex stretching, in this case, is limited since the circulation about a large contour of fluid particles (yet small, still, on the wave scale), and hence the net vorticity inside it, must not change in time, if the average depth within the contour is always the same; again it is the net change in depth that counts. The intermediate cases, when Taylor columns begin to grow above obstacles, or alternatively when $U/f \omega L_R \sim 1$, are not well understood, but one hopes that they are the logical bridges between the extremes.

The effect of isolated one-dimensional features can, it turns out, be predicted for highly non-linear motions. Bretherton (private communication) has pointed out that the matching condition for the velocities over a step can be found by applying Kelvin's theorem to a block of fluid straddling the contours of depth, as shown in figure 13. It must be assumed that velocities change very slowly

along the step (in the ξ direction): that is, on the scale comparable to a wavelength. The fluid forced up the slope satisfies

$$(hv)_\eta \approx 0.$$

If a line of particles in deeper water moves a distance normal to the step Y , this is magnified slightly, to $Y(1 + \delta)$ on the other side (Y may be much greater than a , the step width). The circulation, Γ , about a rectangle of length L parallel to the slope, and width $\eta_2 - \eta_1$, is initially

$$(u_1 - u_2)L$$

where the subscripts 2 and 1 refer to the shallow and deep sides, respectively. As the block moves it is distorted by vortex stretching over the step, but as long as $\partial/\partial\xi \ll 1/a$, and with the contour broad enough so that it always spans the step,

$$\frac{d}{dt} \iint (f - u_\eta) dS = 0$$

where dS is a plane horizontal element of fluid. If we ignore the change in f within the contour,

$$d\Gamma/dt = -fLd/dt(\eta_2 - \eta_1),$$

and it follows that

$$\frac{D}{Dt} (u_1 - u_2) + f\delta \frac{dY}{dt} = 0$$

or

$$\frac{D}{Dt} (u_1 - u_2) + f\delta v = 0.$$

For matching to linear Rossby waves in the exterior this vortex sheet is essentially fixed in space, and it brings about the same amount of reflexion as in the case of infinitesimal displacements. The argument also secures the trapped wave solutions in the face of non-linearities.

Without motion, a step in the depth divides two regions of different potential vorticity so that even when fluid from one side infringes greatly on the other it is still in a hostile environment. With a ridge that falls off to the same depth on each side, the potential vorticity at rest is the same everywhere but over the ridge. Slight displacement of this block of fluid due to small-amplitude Rossby waves produces long slope velocities in different directions on the slopes of different sign; this is the double vortex sheet found to be impenetrable if $(k/l)(\delta/\omega)^2 ka \gg 1$. If, however, columns of fluid traverse the ridge completely, one of the vortex sheets is swept downstream, like the 'starting vortex' due to an aerofoil, leaving the other bound to the ridge. This causes a net change in direction of the streamlines as they cross the ridge. The reader may see this behaviour demonstrated in the computer solutions of Kasahara (1966).

Large, periodic displacements may again be treated with Kelvin's theorem. If the contour extends far outside the ridge, the same assumption that $\partial/\partial\xi \ll 1/a$ means that the normal velocities before and after the ridge are the same,

$$v_1 = v_2,$$

and hence

$$\frac{D}{Dt} (u_2 - u_1) = 0,$$

and there is no reflexion at all in this limit. The influence of the ridge will, in general, depend on the fraction of the block of fluid initially over the ridge that is moved off it by the wave motion.

One-dimensional topography, $h = h(y)$ only, is special, in that geostrophic flow $\psi(f/h)$ cannot cross it. This is a linear (but singular) steady flow limit whereas the present case is fundamentally non-linear.

Excitation of trapped waves

The wind stress, τ , at the surface is usually taken to act as a body force on the interior fluid, due to the convergence in quasi-steady, surface Ekman layers. Vorticity is thus produced at a rate $|\nabla \wedge \tau|/\rho h$. The frequent appearance of these body forces in the *momentum* balance is unjustified, however, and with varying depth leads to a large, false forcing term, $\tau/\rho \wedge \nabla(1/h)$; in the absence of lateral boundaries a stress constant in space will merely drive the Ekman layer at right angles, leaving the interior at rest.

The trapped modes over steps and ridges cannot be forced at resonance by Rossby waves in the exterior since the dispersion relations are mutually exclusive. If, however, there is a periodic wind stress 'tweak' forcing equation (3.1) at $\eta = \eta_0$,

$$\frac{|\nabla \wedge \tau|}{\rho H} = \Delta(\eta - \eta_0) \exp\{i(k\xi - \omega_0 f_0 t)\},$$

near an abrupt step, where $\delta = [h(0-)/h(0+)] - 1$, and $\Delta(\chi)$ is the delta function, the response $\bar{\psi}$ has a magnitude (at $\eta = 0$)

$$G(\xi, 0, \eta_0) = \frac{i \exp\{-k\eta_0\}}{kf_0(2+\delta)[\omega_0 - \delta/(2+\delta)]} \exp\{i(k\xi - \omega_0 f_0 t)\} \quad (k, \eta_0 > 0).$$

The motion over the step drops exponentially with the distance of the 'storm', and linearly with the separation of natural and forcing frequencies. The ratio of $G(\xi, 0, \eta_0)$ to the amplitude with $\delta = 0$ is just $2[(\delta/\omega_0) - (2+\delta)]^{-1}$. The asymmetry of the response under a reverse in the direction of motion of the forcing agent ($\omega_0 \rightarrow -\omega_0$) is expected from the free wave results. In the absence of resonant amplification ($k\eta_0 \gg 1$) the interior velocities are of order $|\tau|/\rho H f_0 \omega_0$.

Applications

In summary of the wave guide nature of bunched contours, the largest-scale and highest-frequency modes have group velocity to the 'right' (if one faces shallow water), and their fringes may extend far from the topography. The modes with more structure have lower frequency and are confined near the topography. With rapid variation along the contours the group velocity points to the 'left'; with gradual variation, to the 'right'. A large-scale wind stress should thus excite high frequencies to its 'right' and low frequencies to its 'left'. Motions at vanishingly low frequencies propagate away from a source faster than the rest, and so jet-like wakes extending along the contours to the 'left' of a moving disturbance may be expected. The Mid-Atlantic Ridge and its extensions, alone, provide many such wave guides (the longest waves over a ridge will

be affected by cancellation between the vortex sheets, but the shorter ones will not).

The continental shelf is a likely site of trapped waves. They should dominate the quasigeostrophic spectrum, since few Rossby waves will penetrate the rim. The mode found by Robinson (1964) and Mysak (1967) has a scale along the coast, k^{-1} , much greater than the width of the shelf, a (the essential physics was given by Reid (1956)). It resembles the slope (or sloping-step) waves found in §2. The appropriate limit is $l \gg k$ for which $\omega \sim ka$, since $l \sim 1/a$, $M \sim 1/a$. This wave guide is, again, non-dispersive, the energy travelling a distance $\sim 2\pi a$ in a day.

The present results show that in addition there is a wave of much higher frequency ($\omega \approx \delta/(2+\delta) \approx 1$) and also a family of dispersive waves with $\omega \lesssim ka$ over the steep rim of the shelf. When k^{-1} is extremely large the high-frequency motion becomes a Kelvin wave, since free surface motion is then important. The group velocity of the lower-frequency waves carries them, typically, 2π rim-widths in a day. This should thus be a region of tempestuous motions, and, the smaller their scale, the more they will be correlated with nearby conditions.

Friction and advection will quickly destroy the waves; viscous spin-down should take only a few days, especially in the shallow water over the shelf. The mode found by Robinson is, however, the one quasigeostrophic wave that has been observed with some certainty: see Hamon (1966).

4. Conclusions

The measure of topographic effects on slow oscillations is not δ , but δ/ω (for a step), $\delta R/a \equiv MR$ (slope), or $(\delta/\omega)(ka)^{1/2}$ (narrow ridge). A step acts like a single vortex sheet to the external field, a ridge like a double sheet (recall that with a steady current a ridge produces a *single* vortex sheet). In a smoothly sloping ocean basin of scale L , quasigeostrophic waves will often rely on topography rather than β (when $L/R \ll 1$). Qualitative experiments demonstrating the reflexion off ridges are reported in Rhines (1967).

Short waves ($ka \gg 1$) trapped along bunched contours of f/h should be an especially vigorous feature of the ocean, for their group velocity is small. If $ka \ll 1$ energy may be guided more efficiently (at velocities $\lesssim f_0 \delta a/\pi$ for a step). The limit $\omega \rightarrow 0$ shows that blocking effects propagate rapidly to the left, looking up the slope.

It was predicted that few Rossby waves will penetrate the continental rise. Some confirmation is given by Wunsch (1967), reported at the time of this writing. He found that Rossby waves, generated by fortnightly and monthly tidal potentials, are somewhat evident at island stations, yet in six records at the American west coast just one peak appeared above the noise. The result is not in conflict with the active low-frequency spectra generally seen at coastal tide gauges. Topographic waves generated locally, when fluid is forced at the coast by wind stress, could account for these observations.

Some effects neglected here will be treated in succeeding papers. The first, density stratification, sometimes lessens the influence of topography, and a new

kind of trapped wave appears. We have ignored any curvature of the depth contours: a study with circular contours shows that trapped waves exist about an arbitrarily small island. At the same time Rossby waves easily penetrate to the shore, which helps to explain Wunsch's successful observations. Waves over a continuously rough bottom are a more difficult problem, but generalizations of the trapped waves found here do exist.

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